ON SOME SECOND ORDER NONLOCAL FUNCTIONAL AND ORDINARY BOUNDARY VALUE PROBLEMS

P. CH. TSAMATOS

ABSTRACT. In this paper we prove the existence of multiple positive solutions for second order nonlinear functional and ordinary nonlocal boundary value problems. The results are obtained by using a fixed point theorem on a Banach space, ordered by an appropriate cone.

1. INTRODUCTION

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}^+ =: [0, +\infty)$ and $I =: [0, 1]$. Also, let $q \in [0, 1)$ and $J := [-q, 0]$. For every closed interval $B \subseteq J \cup I$ we denote by $C(B)$ the Banach space of all continuous real functions $\psi : B \to \mathbb{R}$ endowed with the usual sup-norm

$$\|\psi\|_B := \sup\{|\psi(s)| : s \in B\}.$$

Also, we define the set $C^+(B)$ as follows

$$C^+(B) := \{\psi \in C(B) : \psi \geq 0\}.$$

If $x \in C(J \cup I)$ and $t \in I$, then we denote by x_t the element of $C(J)$ defined by

$$x_t(s) = x(t + s), \quad s \in J.$$

Now, consider the equation

$$x''(t) + f(t, x_t) = 0, \quad t \in I, \tag{1.1}$$

along with the boundary conditions

$$x_0 = \phi, \tag{1.2}$$

and

$$x'(1) = \int_0^1 x'(s)dg(s), \tag{1.3}$$

where $f : \mathbb{R}^+ \times C^+(J) \to \mathbb{R}^+$ and $\phi : J \to \mathbb{R}^+$ are continuous functions, $g : I \to \mathbb{R}$ is a nondecreasing function, such that $g(0) = 0$ and $1 - g(1) > 0$.

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The problem of existence of positive solutions for boundary value problems for second order differential equations which involve a nonlocal condition like (1.3) has been treated recently by Karakostas and Tsamatos [8-11] and Tsamatos [20]. Moreover, boundary value problems with integral boundary conditions for second order differential equations with retarded arguments is the subject of the papers [12] and [17]. In the recent years an increasing interest is also observed for boundary value problems concerning functional differential equations (see [5,6] and the references therein). Fixed point theorems on Banach spaces ordered by appropriate cones is usually the tool for proving multiple positive solutions for boundary value problems. The famous Guo - Krasnoselskii fixed point theorem [13,22] seems to be used in the largest part of the papers on this subject. Also the well known Leggett-Williams fixed point theorem [14] and some recent generalizations of it are used in proving multiple positive solutions for various types of boundary value problems.

For a detailed exposition of the theory of functional differential equations, like (1.1), the reader can refer to the books due to Hale and Lunel [4] and Azbelev *et all.* [3].

In this paper, we choose to use a fixed point theorem, on an ordered by cones Banach space, due to Avery and Henderson [1] (see also [15,19])which, apart from guarantying the existence of two positive solutions, has the advantage to offer some additional information on these solutions. In our results, the values of these solutions at three given points of their domain are upper or lower bounded by a-priori given constants. We note that this fixed point theorem was used recently in several papers (see [2,5,15,16,18,19] and the references therein).

Since the results we present are new even in the ordinary case, we mention them for this case too, underlining the necessary adjustments that have to be made to the hypothesis referring to the functional case.

The paper is organized as follows. In section 2 we present the definitions and the lemmas we are going to use, as well as the fixed point theorem, on which we base our results. In section 3, we present the new results for the functional case and then in section 4 the results for the ordinary case. Finally, in section 5 we give some applications of our results.

2. PRELIMINARIES AND SOME BASIC LEMMAS

Definition. *A function* $x \in C(J \cup I)$ *is a solution of the boundary value problem* $(1.1) - (1.3)$ *if* x *satisfies equation* (1.1), *the boundary condition* (1.3) *and, moreover* $x|J = \phi$.

Lemma 2.1. *A function* $x \in C(J \cup I)$ *is a solution of the boundary value problem* $(1.1) - (1.3)$ *if and only if* x *is a fixed point of the operator* $A : C(J \cup I) \to C(J \cup I)$, *with*

$$Ax(t) = \begin{cases} \phi(t), \ \ t \in J \\ \phi(0) + \zeta t \int_0^1 \int_s^1 f(r, x_r) dr dg(s) + \int_0^t \int_s^1 f(r, x_r) dr ds, \ \ t \in I, \end{cases}$$

where $\zeta := \frac{1}{1-g(1)}$.

Proof. Suppose that x is a solution of the boundary value problem $(1.1) - (1.3)$. Then, obviously, $x|J = \phi$. Moreover, by integrating (1.1) we get

$$x'(t) = x'(1) + \int_t^1 f(s, x_s) ds, \ \ t \in I. \tag{2.1}$$

Also from (1.3) and (2.1) we have

$$\begin{aligned} x'(1) &= \int_0^1 x'(s)dg(s) \\ &= \int_0^1 \left(x'(1) + \int_s^1 f(r, x_r)dr\right) dg(s) \\ &= x'(1)\int_0^1 dg(s) + \int_0^1 \int_s^1 f(r, x_r)drdg(s). \end{aligned}$$

Therefore

$$x'(1) = \zeta \int_0^1 \int_s^1 f(r, x_r)drdg(s). \tag{2.2}$$

Combining (2.1) and (2.2) we conclude that

$$x'(t) = \zeta \int_0^1 \int_s^1 f(r, x_r)drdg(s) + \int_t^1 f(s, x_s)ds, \ \ t \in I,$$

which by integration from 0 to t, $t \in I$ gives

$$\begin{aligned} x(t) &= x(0) + \zeta t \int_0^1 \int_s^1 f(r, x_r)drdg(s) + \int_0^t \int_s^1 f(r, x_r)drds \\ &= \phi(0) + \zeta t \int_0^1 \int_s^1 f(r, x_r)drdg(s) + \int_0^t \int_s^1 f(r, x_r)drds. \end{aligned}$$

The above step gives that, if x is a solution of the boundary value problem (1.1) – (1.3), then $x = Ax$.

For the inverse, suppose that x is a fixed point of the operator A. Then, obviously, $\phi(t) = x(t) = x(0 + t) = x_0(t)$ for $t \in J$. Also from the form of A we have

$$x'(t) = \zeta \int_0^1 \int_s^1 f(r, x_r)drdg(s) + \int_t^1 f(r, x_r)dr, \ \ t \in I. \tag{2.3}$$

Therefore

$$x'(1) = \zeta \int_0^1 \int_s^1 f(r, x_r)drdg(s). \tag{2.4}$$

Then from (2.3), (2.4) and the fact that $\zeta := \frac{1}{1-g(1)}$, $g(0) = 0$, we get

$$x'(t) = x'(1) + \int_t^1 f(s, x_s)ds, \ \ t \in I. \tag{2.5}$$

Using (2.4) and (2.5) we have

$$\begin{aligned} x'(1) &= x'(1)g(1) + \int_0^1 \int_s^1 f(r, x_r)drdg(s) \\ &= x'(1)\int_0^1 dg(s) + \int_0^1 \int_s^1 f(r, x_r)drdg(s) \\ &= \int_0^1 \left(x'(1) + \int_s^1 f(r, x_r)dr\right) dg(s) \\ &= \int_0^1 x'(s)dg(s). \end{aligned}$$

Finally, from (2.5) we have

$$x'(t) = x'(1) - \int_1^t f(s, x_s)ds, \ t \in I$$

and so

$$x''(t) + f(t, x_t) = 0, \ t \in I.$$

The proof is complete. $\square$

The following lemma can be found in [21].

Lemma 2.2. *Let function $x \in C(I)$ be concave and non negative and $\xi \in (0, \frac{1}{2})$. Then*

$$(1) \ x(t) \geq \begin{cases} \frac{\|x\|t}{\sigma}, & 0 \leq t \leq \sigma, \\ \|x\|\frac{1-t}{1-\sigma}, & \sigma \leq t \leq 1, \end{cases} \quad \text{if } 0 < \sigma < 1,$$

$$(2) \ x(t) \geq \|x\|t, \ 0 \leq t \leq 1, \ \text{if } \sigma = 1,$$

$$(3) \ x(t) \geq \|x\|(1-t), \ 0 \leq t \leq 1, \ \text{if } \sigma = 0,$$

$$(4) \ x(t) \geq \xi\|x\|, \ \text{for all } t \in [\xi, 1-\xi],$$

where $\|x\| := \sup\{|x(t)| : 0 \leq t \leq 1\}$ and $\sigma \in [0, 1]$ such that $x(\sigma) = \|x\|$.

The results proved in this paper are based on the following Theorem 2.6 due to R. I. Avery and J. Henderson [1] (see also [15] and [19]). As we mentioned in the introduction, this theorem ensures that our boundary value problem (1.1) – (1.3) has at least two distinct positive solutions and, moreover, for each of these solutions, we have an upper bound at some specific point of its domain and a lower bound at some other specific point of its domain. Also, both solutions are concave and nondecreasing on I. In order to apply this theorem some definitions are necessary.

Definition 2.3. *Let $\mathbb{E}$ be a real Banach space. A cone in $\mathbb{E}$ is a nonempty, closed set $\mathbb{P} \subset \mathbb{E}$ such that*

(*i*) $\kappa u + \lambda v \in \mathbb{P}$ *for all* $u, v \in \mathbb{P}$ *and all* $\kappa, \lambda \geq 0$

(*ii*) $u, -u \in \mathbb{P}$ *implies* $u = 0$.

Definition 2.4. *Let $\mathbb{P}$ be a cone in a real Banach space $\mathbb{B}$. A functional $\psi : \mathbb{P} \longrightarrow \mathbb{B}$ is said to be increasing on $\mathbb{P}$ if $\psi(x) \leq \psi(y)$, for any $x, y \in \mathbb{P}$ with $x \leq y$, where $\leq$ is the partial ordering induced to the Banach space by the cone $\mathbb{P}$, i.e.*

$$x \leq y \ \text{if and only if} \ y - x \in \mathbb{P}.$$

Definition 2.5. *Let ψ be a nonnegative functional on a cone $\mathbb{P}$. For each $d > 0$ we denote by $\mathbb{P}(\psi, d)$ the set*

$$\mathbb{P}(\psi, d) := \{x \in \mathbb{P} : \psi(x) < d\}.$$

Theorem 2.6. *Let $\mathbb{P}$ be a cone in a real Banach space $\mathbb{E}$. Let α and γ be increasing, nonnegative, continuous functionals on $\mathbb{P}$, and let θ be a nonnegative functional on $\mathbb{P}$ with $\theta(0) = 0$ such that, for some $c > 0$ and $\Theta > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \quad \textit{and} \quad \|x\| \leq \Theta\gamma(x),$$

for all $x \in \overline{\mathbb{P}(\gamma, c)}$. Suppose there exists a completely continuous operator $A : \overline{\mathbb{P}(\gamma, c)} \to \mathbb{P}$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x), \quad \textit{for} \quad 0 \leq \lambda \leq 1 \quad \textit{and} \quad x \in \vartheta\mathbb{P}(\theta, b),$$

and

(*i*) $\gamma(Ax) > c$, *for all* $x \in \partial\mathbb{P}(\gamma, c)$,

(*ii*) $\theta(Ax) < b$, *for all* $x \in \partial\mathbb{P}(\theta, b)$,

(*iii*) $\mathbb{P}(\alpha, a) \neq \emptyset$, *and* $\alpha(Ax) > a$, *for all* $x \in \partial\mathbb{P}(\alpha, a)$,

or

(*i'*) $\gamma(Ax) < c$, *for all* $x \in \partial\mathbb{P}(\gamma, c)$,

(*ii'*) $\theta(Ax) > b$, *for all* $x \in \partial\mathbb{P}(\theta, b)$,

(*iii'*) $\mathbb{P}(\alpha, a) \neq \emptyset$, *and* $\alpha(Ax) < a$, *for all* $x \in \partial\mathbb{P}(\alpha, a)$.

Then A has at least two fixed points x_1 and x_2 belonging to $\overline{\mathbb{P}(\gamma, c)}$ such that

$$a < \alpha(x_1), \quad \textit{with} \quad \theta(x_1) < b,$$

and

$$b < \theta(x_2), \quad \textit{with} \quad \gamma(x_2) < c.$$

3. Main Results

Define the set

$$\mathbb{K} := \{x \in C(J \cup I) : x(t) \geq 0, \;\; t \in J \cup I, \;\; x/I \text{ is concave and nondecreasing}\},$$

which is a cone in $C(J \cup I)$. Also let

$$0 < r_1 \leq r_2 \leq r_3 \leq 1$$

and consider the following functionals

$$\gamma(x) = x(r_1), \;\; x \in \mathbb{K},$$

$$\theta(x) = x(r_2), \;\; x \in \mathbb{K}$$

and

$$\alpha(x) = x(r_3), \;\; x \in \mathbb{K}.$$

It is easy to see that α, γ are nonnegative, increasing and continuous functionals on $\mathbb{K}$, θ is nonnegative on $\mathbb{K}$ and $\theta(0) = 0$. Also, it is straightforward that

$$\gamma(x) \leq \theta(x) \leq \alpha(x), \tag{3.1}$$

since $x \in \mathbb{K}$ is nondecreasing on I. Furthermore, for any $x \in \mathbb{K}$, by Lemma 2.2 (inequality (2)), we have

$$\gamma(x) = x(r_1) \geq r_1 \|x\|_I.$$

So

$$\|x\|_I \leq \frac{1}{r_1}\gamma(x), \quad x \in \mathbb{K}. \tag{3.2}$$

Additionally, by the definition of θ it is obvious that

$$\theta(\lambda x) = \lambda\theta(x), \quad 0 \leq \lambda \leq 1, \quad x \in \mathbb{K}.$$

Now, if $D \subset I$, consider the functions $H : C(I) \to C(I)$ and $H_D : C(I) \to C(I)$ by

$$(Hz)(s) := \int_s^1 z(r)dr, \quad s \in I$$

and

$$(H_D z)(s) := \int_{D\cap[s,1]} z(r)dr, \quad s \in I.$$

At this point, we state the following assumptions:

(H_1) There exist $M > 0$, continuous function $u : I \to \mathbb{R}^+$ and nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \leq u(t)L(\|y\|_J), \quad t \in I, \quad y \in C^+(J)$$

and also

$$\phi(0) + L(M)\left(\zeta r_2 \int_0^1 (Hu)(s)dg(s) + \int_0^{r_2} (Hu)(s)ds\right) < Mr_2.$$

(H_2) There exist a constant $\delta \in (0, 1)$ and functions $\tau : I \to [0, q]$, continuous $v : I \to \mathbb{R}^+$ and nondecreasing $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \geq v(t)w(y(-\tau(t))), \quad t \in X, \quad y \in \{h \in C^+(J) : \|h\|_J < M\},$$

where

$$X := \{t \in I : \delta \leq t - \tau(t) \leq 1\},$$

$\sup\{v(t) : t \in X\} > 0$, $meas(X \cap [s, 1)) > 0$ for all $s \in [0, 1)$ and M is defined in (H_1).

(H_3) There exist $\rho_1, \rho_3 > 0$ such that

$$\frac{\rho_i}{\delta} < \phi(0) + w(\rho_i)\left(r_i\zeta \int_0^1 (H_X v)(s)dg(s) + \int_0^{r_i} (H_X v)(s)ds\right), \quad i = 1, 3,$$

and

$$\frac{\rho_3}{\delta} < Mr_2 < \frac{\rho_1}{\delta}.$$

Notice that if $\phi(0) \neq 0$, then these ρ_1, ρ_3 always exist.

Remark. *It easy to see that* $\sup\{v(t) : t \in X\} > 0$ *and* $meas(X \cap [s, 1)) > 0$, $s \in [0, 1)$ *in the assumption* (H_2), *imply that* $\int_0^1 (H_X v)(s)dg(s) > 0$ *and* $\int_0^{r_i} (H_X v)(s)ds > 0$, $i = 1, 3$.

Theorem 3.1. *Suppose that assumptions* (H_1)–(H_3) *hold and furthermore* $\|\phi\|_J \leq M$. *Then the boundary value problem* $(1.1) - (1.3)$ *has at least two concave and nondecreasing on* I *and positive on* $J \cup I$ *solutions* x_1, x_2 *such that* $x_1(r_3) > \frac{\rho_3}{\delta}$, $x_1(r_2) < Mr_2$, $x_2(r_2) > Mr_2$ *and* $x_2(r_1) < \frac{\rho_1}{\delta}$.

Proof. First of all, we observe that, because of (H_1), $f(t, \cdot)$ maps bounded sets into bounded sets. Therefore A is a completely continuous operator.

Now we set $a = \frac{\rho_3}{\delta}$, $b = Mr_2$, $c = \frac{\rho_1}{\delta}$ and we consider a $x \in \overline{\mathbb{K}(\gamma, c)}$. Then since $\zeta > 0$ and $f(t, x_t) \geq 0$ for every $t \in I$, we get that $Ax(t) \geq 0$, $t \in I$. Also $Ax(t) = \phi(t) \geq 0$, $t \in J$. Thus $Ax(t) \geq 0$, $t \in J \cup I$. Moreover, $(Ax)''(t) = -f(t, x_t) \leq 0$, which means that Ax is concave on I. Also it is clear that $(Ax)'(t) \geq 0$ for $t \in I$. So $A : \overline{\mathbb{K}(\gamma, c)} \to \mathbb{K}$.

Now let $x \in \partial\mathbb{K}(\gamma, c)$. Then $\gamma(x) = x(r_1) = c$ and so $\|x\|_I \geq c$. Having in mind assumption (H_2), we get

$$\begin{aligned}
\gamma(Ax) = Ax(r_1) &= \phi(0) + \zeta r_1 \int_0^1 \int_s^1 f(r, x_r)drdg(s) + \int_0^{r_1} \int_s^1 f(r, x_r)drds \\
&\geq \phi(0) + \zeta r_1 \int_0^1 \int_{X\cap[s,1]} f(r, x_r)drdg(s) + \int_0^{r_1} \int_{X\cap[s,1]} f(r, x_r)drds \\
&\geq \phi(0) + \zeta r_1 \int_0^1 \int_{X\cap[s,1]} v(r)w(x_r(-\tau(r)))drdg(s) \\
&\quad + \int_0^{r_1} \int_{X\cap[s,1]} v(r)w(x_r(-\tau(r)))drds \\
&= \phi(0) + \zeta r_1 \int_0^1 \int_{X\cap[s,1]} v(r)w(x(r - \tau(r)))drdg(s) \\
&\quad + \int_0^{r_1} \int_{X\cap[s,1]} v(r)w(x(r - \tau(r)))drds \\
&\geq \phi(0) + \zeta r_1 \int_0^1 \int_{X\cap[s,1]} v(r)w(x(\delta))drdg(s) \\
&\quad + \int_0^{r_1} \int_{X\cap[s,1]} v(r)w(x(\delta))drds.
\end{aligned}$$

Additionally, by assumption (H_3) and inequality 2 of Lemma 2.2, we have

$$\begin{aligned}
\gamma(Ax) &\geq \phi(0) + w(\delta\|x\|_I) \left(r_1\zeta \int_0^1 (H_X v)(s)dg(s) + \int_0^{r_1} (H_X v)(s)ds \right) \\
&\geq \phi(0) + w(\delta c) \left(r_1\zeta \int_0^1 (H_X v)(s)dg(s) + \int_0^{r_1} (H_X v)(s)ds \right) \\
&= \phi(0) + w(\rho_1) \left(r_1\zeta \int_0^1 (H_X v)(s)dg(s) + \int_0^{r_1} (H_X v)(s)ds \right) \\
&> \frac{\rho_1}{\delta} = c.
\end{aligned}$$

This means that condition (i) of Theorem 2.5 is satisfied.

Now let $x \in \partial\mathbb{K}(\theta, b)$. Then $\theta(x) = x(r_2) = b$ and so by inequality 2 of Lemma 2.2 we get

$$\|x\|_I \le \frac{1}{r_2}x(r_2) = \frac{1}{r_2}\theta(x) = \frac{b}{r_2}.$$

Also we assumed that $\|\phi\|_J \le M = \frac{b}{r_2}$, so $\|x\|_{J\cup I} \le \frac{b}{r_2}$. Now, by (H_1), we have

$$\begin{aligned}\theta(Ax) &= Ax(r_2)\\ &= \phi(0) + \zeta r_2 \int_0^1 \int_s^1 f(r, x_r)drdg(s) + \int_0^{r_2}\int_s^1 f(r, x_r)drds\\ &\le \phi(0) + \zeta r_2 \int_0^1 \int_s^1 u(r)L(\|x_r\|_J)drdg(s) + \int_0^{r_2}\int_s^1 u(r)L(\|x_r\|_J)drds\\ &\le \phi(0) + \zeta r_2 \int_0^1 \int_s^1 u(r)L(\frac{b}{r_2})drdg(s) + \int_0^{r_2}\int_s^1 u(r)L(\frac{b}{r_2})drds\\ &= \phi(0) + L(M)\left(\zeta r_2 \int_0^1 (Hu)(s)dg(s) + \int_0^{r_2}(Hu)(s)ds\right)\\ &< Mr_2 = b.\end{aligned}$$

So condition (ii) of Theorem 2.5 is also satisfied.

Now, define the function $y : J \cup I \to \mathbb{R}$ with $y(t) = \frac{a}{2}$. Then it is obvious that $\alpha(y) = \frac{a}{2} < a$, so $\mathbb{K}(\alpha, a) \neq \emptyset$. Also, for any $x \in \partial\mathbb{K}(\alpha, a)$ we have $\alpha(x) = x(r_3) = a$. Therefore $\|x\|_I \ge a$. Now, having in mind assumption (H_2) and as in the case of the functional γ above, we get

$$\begin{aligned}\alpha(Ax) &= Ax(r_3)\\ &\ge \phi(0) + \zeta r_3 \int_0^1 \int_{X\cap[s,1]} v(r)w(x(\delta))drdg(s)\\ &\quad + \int_0^{r_3}\int_{X\cap[s,1]} v(r)w(x(\delta))drds.\end{aligned}$$

Then, by assumption (H_3) and inequality 2 of Lemma 2.2, we also have

$$\alpha(Ax) \ge \phi(0) + w(\delta a)\left(r_3\zeta \int_0^1 (H_X v)(s)dg(s) + \int_0^{r_3}(H_X v)(s)ds\right) > \frac{\rho_3}{\delta} = a.$$

Consequently, assumption (iii) of Theorem 2.5 is satisfied.

The result can now be obtained by applying Theorem 2.5. □

The above Theorem 3.1 has been obtained by using the requirements $(i) - (iii)$ of Theorem 2.5. Using the requirements $(i') - (iii')$ of the same theorem we can also obtain another existence theorem (Theorem 3.2 below) for our boundary value problem (1.1) – (1.3). For this purpose we need the following assumptions.

$(\widehat{H}_1)$ There exist $M_1, M_3 > 0$, continuous function $u : I \to \mathbb{R}^+$ and nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \le u(t)L(\|y\|_J), \quad t \in I, \quad y \in C^+(J)$$

and also

$$\phi(0) + L(M_i)\left(\zeta r_i \int_0^1 (Hu)(s)dg(s) + \int_0^{r_i} (Hu)(s)ds\right) < M_i r_i, \ \ i = 1, 3.$$

($\widehat{H}_2$) There exist a constant $\delta \in (0,1)$ and functions $\tau : I \to [0,q]$, continuous $v : I \to \mathbb{R}^+$ and nondecreasing $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t,y) \geq v(t)w(y(-\tau(t))), \ \ t \in X, \ \ y \in \{h \in C^+(J) : \|h\|_J < \min\{M_1, M_3\}\},$$

where

$$X := \{t \in I : \delta \leq t - \tau(t) \leq 1\},$$

$\sup\{v(t) : t \in X\} > 0$, $meas(X \cap [s,1]) > 0$ for all $s \in [0,1)$ and M is defined in (H_4).

($\widehat{H}_3$) There exists $\rho > 0$ such that

$$\frac{\rho}{\delta} < \phi(0) + w(\rho)\left(r_2\zeta \int_0^1 (H_X v)(s)dg(s) + \int_0^{r_2} (H_X v)(s)ds\right).$$

Notice that if $\phi(0) \neq 0$, then this ρ always exists.

Theorem 3.2. *Suppose that assumptions* ($\widehat{H}_1$)–($\widehat{H}_3$) *hold and furthermore* $\|\phi\|_J \leq \min\{M_1, M_3\}$.

Then the boundary value problem (1.1) – (1.3) *has at least two concave and nondecreasing on I and positive on $J \cup I$ solutions x_1, x_2 such that* $x_1(r_3) > M_3 r_3$, $x_1(r_2) < \frac{\rho}{\delta}$, $x_2(r_2) > \frac{\rho}{\delta}$ *and* $x_2(r_1) < M_1 r_1$.

Proof. Consider the functionals γ, θ, α as in Theorem 3.1. Our purpose is to prove that requirements $(i'), (ii'), (iii')$ are satisfied.

Let $a = M_3 r_3$, $b = \frac{\rho}{\delta}$, $c = M_1 r_1$ and $x \in \partial\mathbb{K}(\gamma, c)$. Then $\gamma(x) = x(r_1) = c$ and, by Lemma 2.2 (inequality 2), we get

$$\|x\|_I \leq \frac{1}{r_1} x(r_1) = \frac{1}{r_1}\gamma(x) = \frac{c}{r_1}.$$

Also we assumed that $\|\phi\|_J \leq M_1 = \frac{c}{r_1}$, so $\|x\|_{J\cup I} \leq \frac{c}{r_1}$. Then, by ($\widehat{H}_1$) and following the same arguments as in the proof of Theorem 3.1 we can easily prove that

$$\gamma(Ax) < c.$$

So condition (i') of Theorem 2.5 is satisfied.

Now let $x \in \partial\mathbb{K}(\theta, b)$. Then $\theta(x) = x(r_2) = b$, so $\|x\|_I \geq b$. Hence, having in mind assumptions ($\widehat{H}_2$), ($\widehat{H}_3$) and inequality 2 of Lemma 2.2, as in the proof of Theorem 3.1, we can prove that

$$\theta(Ax) > b.$$

This means that condition (ii') of Theorem 2.5 is satisfied.

Now, define the function $y : J \cup I \to \mathbb{R}$ with $y(t) = \frac{a}{2}$. Then it is obvious that $\alpha(y) = \frac{a}{2} < a$, so $\mathbb{K}(\alpha, a) \neq \emptyset$. Also let $x \in \partial\mathbb{K}(\alpha, a)$. Then $\alpha(x) = x(r_3) = a$. So, by Lemma 2.2 (inequality 2) we get

$$\|x\|_I \leq \frac{1}{r_3} x(r_3) = \frac{1}{r_3}\alpha(x) = \frac{a}{r_3}.$$

Also we assumed that $\|\phi\|_J \leq M_3 = \frac{a}{r_3}$, so $\|x\|_{J\cup I} \leq \frac{a}{r_3}$. Now, by $(\widehat{H}_1)$, we can easily prove that

$$\alpha(Ax) < a.$$

Consequently, assumption (iii') of Theorem 2.5 is satisfied and our proof is completed. □

The obtained solutions x_1, x_2 in Theorems 3.1, 3.2 above are all nondecreasing. Thus, in the special case when $r_1 = r_2 = r_3 = 1$, we have that $x_i(r_j) = x_i(1) = \|x_i\|$, $i = 1, 2$, $j = 1, 2, 3$. Therefore, we have the following corollary of Theorems 3.1 and 3.2.

Corollary 3.3. *Suppose that assumptions* $(H_1) - (H_3)$ *(resp.* $(\widehat{H}_1) - (\widehat{H}_3)$*) hold and furthermore* $\|\phi\|_J \leq M$ *(resp.* $\|\phi\|_J \leq \min\{M_1, M_3\}$*). Then the boundary value problem* (1.1) − (1.3) *has at least two concave and nondecreasing on* I *and positive on* $J \cup I$ *solutions* x_1, x_2 *such that*

$$\frac{\rho_3}{\delta} < \|x_1\| < M < \|x_2\| < \frac{\rho_1}{\delta}$$

(resp. $M_3 < \|x_1\| < \frac{\rho}{\delta} < \|x_2\| < M_1$*).*

It is remarkable to observe that this corollary can be also obtained by applying twice the Krasnoselkii's theorem under the same assumptions $(H_1) - (H_3)$ (*resp.* $(\widehat{H}_1) - (\widehat{H}_3)$).

4. THE ORDINARY CASE

In this section we suppose that $q = 0$. Then $J = \{0\}$, so the boundary value problem (1.1) − (1.3) is reformulated as follows

$$x''(t) + f(t, x(t)) = 0, \;\; t \in I, \tag{4.1}$$

$$x(0) = N, \tag{4.2}$$

$$x'(1) = \int_0^1 x'(s)dg(s), \tag{4.3}$$

where $f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $g : I \to \mathbb{R}^+$ is a nondecreasing function, such that $g(0) = 0$, $1 - g(1) > 0$ and $N \in \mathbb{R}^+$. Note that equation (4.1) is equivalent to the following form

$$x''(t) + f(t, x_t(0)) = 0, \;\; t \in I$$

and $C^+(\{0\}) \equiv \mathbb{R}^+$, so $f : \mathbb{R}^+ \times C^+(\{0\}) \to \mathbb{R}^+$.

Now, the analogue of Lemma 2.1 for this case is the following

Lemma 4.1. *A function $x \in C(I)$ is a solution of the boundary value problem (4.1) – (4.3) if and only if x is a fixed point of the operator $\widehat{A} : C(I) \to C(I)$, with*

$$\widehat{A}x(t) = N + \zeta t \int_0^1 \int_s^1 f(r, x(r))drdg(s) + \int_0^t \int_s^1 f(r, x(r))drds, \quad t \in I,$$

where $\zeta := \frac{1}{1-g(1)}$.

Assumptions $(H_1) - (H_3)$ and $(\widehat{H}_1) - (\widehat{H}_3)$, for the special case $q = 0$, are stated as follows:

$(H_1)_0$ There exist $M > 0$, continuous function $u : I \to \mathbb{R}^+$ and nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \le u(t)L(y), \quad t \in I, y \in \mathbb{R}^+$$

and

$$N + L(M) \left(\zeta r_2 \int_0^1 (Hu)(s)dg(s) + \int_0^{r_2} (Hu)(s)ds \right) \le M r_2,$$

where the function H is defined in the previous section.

$(H_2)_0$ There exist $\delta \in (0,1)$ and functions $v : I \to \mathbb{R}^+$ continuous and $w : \mathbb{R}^+ \to \mathbb{R}^+$ nondecreasing such that

$$f(t, y) \ge v(t)w(y), \quad t \in Z := [\delta, 1], \quad y \in [0, M]$$

and $\sup\{v(t) : t \in Z\} > 0$.

$(H_3)_0$ There exist $\rho_1, \rho_3 > 0$ such that

$$\frac{\rho_i}{\delta} \le N + w(\rho_i) \left(r_i \zeta \int_0^1 (H_Z v)(s)dg(s) + \int_0^{r_i} (H_Z v)(s)ds \right), \quad i = 1, 3,$$

where the function H_Z is defined in previous section.

$(\widehat{H}_1)_0$ There exist $M_1, M_3 > 0$, continuous function $u : I \to \mathbb{R}^+$ and nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \le u(t)L(y), \quad t \in I, \quad y \in \mathbb{R}^+$$

and

$$N + L(M_i) \left(\zeta r_i \int_0^1 (Hu)(s)dg(s) + \int_0^{r_i} (Hu)(s)ds \right) < M_i r_i, \quad i = 1, 3$$

where the function H is defined in the previous section.

$(\widehat{H}_2)_0$ There exist a constant $\delta \in (0,1)$, continuous function $v : I \to \mathbb{R}^+$ and nondecreasing function $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \ge v(t)w(y), \quad t \in Z := [\delta, 1], \quad y \in \mathbb{R}^+$$

and $\sup\{v(t) : t \in Z\} > 0$.

$(\widehat{H}_3)_0$ There exists $\rho > 0$ such that

$$\frac{\rho}{\delta} < N + w(\rho) \left(r_2 \zeta \int_0^1 (H_Z v)(s)dg(s) + \int_0^{r_2} (H_Z v)(s)ds \right),$$

where the function H_Z is defined in the previous section.

Therefore, we have the following theorems, which are the analogue of Theorems 3.1 and 3.2 respectively:

Theorem 4.2. *Suppose that assumptions* $(\widehat{H}_1) - (\widehat{H}_3)$ *hold and furthermore* $N \leq M$. *Then the boundary value problem* $(4.1) - (4.3)$ *has at least two concave, nondecreasing and positive on* I *solutions* x_1, x_2 *such that* $x_1(r_3) > \frac{\rho_3}{\delta}$, $x_1(r_2) < Mr_2$, $x_2(r_2) > Mr_2$ *and* $x_2(r_1) < \frac{\rho_1}{\delta}$.

Theorem 4.3. *Suppose that assumptions* $(\widehat{H}_1)_0 - (\widehat{H}_3)_0$ *hold and furthermore* $N \leq \min\{M_1, M_3\}$. *Then the boundary value problem* $(4.1) - (4.3)$ *has at least two concave, nondecreasing and positive on* I *solutions* x_1, x_2 *such that* $x_1(r_3) > M_3 r_3$, $x_1(r_2) < \frac{\rho}{\delta}$, $x_2(r_2) > \frac{\rho}{\delta}$ *and* $x_2(r_1) < M_1 r_1$.

Also, the following corollary corresponds to Corollary 3.3.

Corollary 4.4. *Suppose that assumptions* $(H_1)_0 - (H_3)_0$ *(resp.* $(\widehat{H}_1)_0 - (\widehat{H}_3)_0$*) hold and furthermore* $N \leq M$ *(resp.* $N \leq \min\{M_1, M_3\}$*). Then the boundary value problem* $(4.1) - (4.3)$ *has at least two concave, nondecreasing and positive on* I *solutions* x_1, x_2 *such that*

$$\frac{\rho_3}{\delta} < \|x_1\| < M < \|x_2\| < \frac{\rho_1}{\delta}$$

(resp. $M_3 < \|x_1\| < \frac{\rho}{\delta} < \|x_2\| < M_1$*).*

5. APPLICATIONS

1. Consider the boundary value problem

$$x''(t) + \left(x(t - \frac{1}{2}) - \frac{4}{5}\right)^5 + 1 = 0, \ t \in I := [0, 1] \tag{5.1}$$

$$x_0(t) = \phi(t) := t^2, \ t \in J := [-\frac{1}{2}, 0] \tag{5.2}$$

and

$$x'(1) = \int_0^1 x'(s) dg(s), \tag{5.3}$$

where $g(t) = \frac{1}{2}t$, $t \in I$.

Obviously, $f(t, y) := (y - \frac{4}{5})^5 + 1$ and ϕ is nonnegative on $\mathbb{R}^+ \times C^+(J)$,$\phi$ is nonnegative on $\mathbb{R}^+$ and g is nondecreasing, with $g(0) = 0$ and $1 - g(1) = \frac{1}{2} > 0$. Set $r_1 = \frac{2}{5}$, $r_2 = \frac{3}{5}$ and $r_3 = \frac{4}{5}$. Define $L(t) = (t - \frac{4}{5})^5 + 1$, $t \in \mathbb{R}^+$, and $u(t) = 1$, $t \in I$. Since inequality

$$L(M) < \frac{5}{6} M$$

holds for $M = \frac{7}{5}$, assumption (H_1) is satisfied.

Additionally, set $\delta = \frac{1}{4}$, $\tau(t) = \frac{1}{2}$, $t \in I$, $v(t) = 1$, $t \in I$ and $w(t) = (t - \frac{4}{5})^5 + 1$, $t \in \mathbb{R}^+$. Then, $X = [\frac{3}{4}, 1]$ and the inequalities in assumption (H_3) take the forms

$$w(\rho_1) > \frac{64}{3} \rho_1 \text{ and } w(\rho_3) > \frac{32}{3} \rho_3,$$

which are satisfied for $\rho_1 = 6$ and $\rho_3 = \frac{1}{20}$. Finally, it is obvious that $\|\phi\|_J \leq \frac{7}{5}$, so we can apply Theorem 3.1 to get that the boundary value problem (5.1) – (5.3) has at least two concave and nondecreasing on I and positive on $J \cup I$ solutions x_1, x_2, such that

$$x_1(\frac{4}{5}) > \frac{1}{5},\ x_1(\frac{3}{5}) < \frac{21}{25},\ x_2(\frac{3}{5}) > \frac{21}{25} \text{ and } x_2(\frac{2}{5}) < 24.$$

2. Once again, consider the boundary value problem (5.1) – (5.3). As mentioned in Application 1, $f(t,y) := (y - \frac{4}{5})^5 + 1$ is nonnegative on $\mathbb{R}^+ \times C^+(J)$, ϕ is nonnegative on $\mathbb{R}^+$ and g is nondecreasing, with $g(0) = 0$ and $1 - g(1) = \frac{1}{2} > 0$.

Having in mind Corollary 3.3, we set $r_1 = r_2 = r_3 = 1$. Define $L(t) = (t - \frac{4}{5})^5 + 1$, $t \in \mathbb{R}^+$, and $u(t) = 1$, $t \in I$. Since inequality

$$L(M) < M$$

holds for $M = \frac{11}{10}$, assumption (H_1) is satisfied.

Additionally, set $\delta = \frac{1}{4}$, $\tau(t) = \frac{1}{2}$, $t \in I$, $v(t) = 1$, $t \in I$ and $w(t) = (t - \frac{4}{5})^5 + 1$, $t \in \mathbb{R}^+$. Then, $X = [\frac{3}{4}, 1]$ and the inequalities in assumption (H_3) take the forms

$$w(\rho_1) > \frac{64}{7}\rho_1 \text{ and } w(\rho_3) > \frac{64}{7}\rho_3,$$

which are satisfied for $\rho_1 = \frac{27}{10}$ and $\rho_3 = \frac{1}{12}$. Finally, it is obvious that $\|\phi\|_J \leq \frac{11}{10}$, so we can apply Corollary 3.3 to get that the boundary value problem (5.1) – (5.3) has at least two concave and nondecreasing on I and positive on $J \cup I$ solutions x_1, x_2, such that

$$\frac{1}{3} < \|x_1\| < \frac{11}{10} < \|x_2\| < \frac{54}{5}.$$

3. Consider the boundary value problem

$$x''(t) + 8\arctan(10x(t) - 14) + 12 = 0,\ \ t \in I := [0,1] \tag{5.4}$$

$$x(0) = 0, \tag{5.5}$$

and

$$x'(1) = \int_0^1 x'(s)dg(s), \tag{5.6}$$

where $g(t) = \frac{1}{4}t$, $t \in I$.

Obviously, $f(t,y) := 8\arctan(10y - 14) + 12$ is positive on $\mathbb{R}^+ \times \mathbb{R}^+$ and g is nondecreasing, with $g(0) = 0$ and $1 - g(1) = \frac{3}{4} > 0$. Set $r_1 = \frac{1}{4}$, $r_2 = \frac{1}{2}$ and $r_3 = \frac{3}{4}$. Define $L(t) = 8\arctan(10t - 14) + 12$, $t \in \mathbb{R}^+$, and $u(t) = 1$, $t \in I$. Since inequalities

$$L(M_1) < \frac{24}{25}M_1 \text{ and } L(M_3) < \frac{24}{19}M_3$$

hold for $M_1 = 30$ and $M_3 = \frac{1}{100}$, assumption $(\widehat{H}_1)_0$ is satisfied.

Additionally, set $\delta = \frac{1}{2}$, $v(t) = 1$, $t \in I$ and $w(t) = 8\arctan(10t - 14) + 12$, $t \in \mathbb{R}^+$. Then, $Z = [\frac{1}{2}, 1]$ and assumption $(\widehat{H}_3)_0$ takes the form

$$w(\rho) > \frac{96}{15}\rho,$$

which is satisfied for $\rho = \frac{7}{5}$. Finally, it is obvious that $N = 0 \leq \frac{1}{100} = \min\{M_1, M_3\}$, so we can apply Theorem 4.3 to get that the boundary value problem (5.4) – (5.6) has at least two concave and nondecreasing and positive on I solutions x_1, x_2, such that

$$x_1(\frac{3}{4}) > \frac{3}{400},\ x_1(\frac{1}{2}) < \frac{14}{5},\ x_2(\frac{1}{2}) > \frac{14}{5} \text{ and } x_2(\frac{1}{4}) < \frac{15}{2}.$$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOANNINA, 451 10 IOANNINA, GREECE
E-mail address: `ptsamato@cc.uoi.gr`